# Lemmas and Proofs for the Ergodic Sum-Rate of Proportional Fair Scheduling

David Morales-Jimenez and Angel Lozano

**Abstract**

This technical note is a supportive document which provides the lemmas and proofs for the paper entitled "Ergodic sum-rate of proportional fair scheduling with multiple antennas", submitted to the IEEE International Symposium on Information Theory 2013 (ISIT 2013).

## I. Introduction

The paper [1] deals with the ergodic sum-rate of a proportional fair scheduler (PFS) in wireless systems where both the base stations and the user terminals are equipped with multiantenna transceivers. The scheduling process is allowed to operate jointly over multiple parallel subchannels, e.g., those created by frequency-division multiaccess/multiplexing. Exact expressions are derived for arbitrary numbers of users and antennas, and arbitrary fading distributions. These results are also specialized to Rayleigh fading and, further, to the low- and high-power regimes. New, more informative expressions are as well derived for the regime of large numbers of users.

Two lemmas support the derivations and results in [1]. This technical note provides these lemmas along with their proofs.

## II. Lemmas

*Lemma 1:* Let the normalized SNRs for all users $(\tilde{\rho}_1, \dots, \tilde{\rho}_{K_\mathrm{t}})$ be i.i.d. RVs with CDF given by

$$F_{\tilde{\rho}}(\xi) = \left(\frac{1}{\Gamma(\nu)}\gamma(\nu, \xi)\right)^{\delta}, \tag{1}$$

where $\gamma(\cdot, \cdot)$ is the lower incomplete gamma function, and the parameters $\delta$ and $\nu$ depend on the multiantenna scheme, as specified in [1, Table I]. Then, the ergodic rate for subchannel $i$ if user $k$ is selected for transmission is

$$E[\mathcal{I}_{i,k}] = -\log_2 e \sum_{r=0}^{i} \alpha_{i,r} \sum_{n=1}^{\delta(K_\mathrm{t}-i+r)} \binom{\delta(K_\mathrm{t}-i+r)}{n} (-1)^n \, e^{n/\rho_k} \sum_{b=0}^{(\nu-1)n} \beta_{n,b}^{(\nu)} \frac{b!}{\rho_k^b} \Gamma\left(-b, \frac{n}{\rho_k}\right) \tag{2}$$

where $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function, $\alpha_{i,r}$ are the combinatorial coefficients defined in [1, (12)], and

$$\beta_{n,b}^{(1)} = 1$$
$$\beta_{n,b}^{(2)} = \frac{n!}{b!\,(n-b)!}$$
$$\beta_{n,b}^{(\nu)} = \sum_{i_\nu=0}^{\lfloor\frac{b}{\nu-1}\rfloor} \sum_{i_{\nu-1}=0}^{\lfloor\frac{b-(\nu-1)i_\nu}{\nu-2}\rfloor} \cdots \sum_{i_3=0}^{\lfloor\frac{b-3i_4-\ldots-(\nu-1)i_\nu}{2}\rfloor}$$
$$\frac{n!}{\left(b-\sum_{r=3}^{\nu}(r-1)i_r\right)!\,\left(n-b+\sum_{r=3}^{\nu}(r-2)i_r\right)!\,\prod_{r=3}^{\nu}(r-1)!^{i_r} i_r!}, \quad \nu > 2. \tag{3}$$

*Proof:* See Appendix A. ■

*Lemma 2:* The ergodic sum-rate expands as $C = \dot{C}(0)\,\rho + \frac{\ddot{C}(0)}{2}\rho^2 + o(\rho^2)$ with

$$\dot{C}(0) = \frac{-\log_2 e}{K_\mathrm{a}}\mathcal{F}_0 \tag{4}$$

$$\ddot{C}(0) = \frac{2\log_2 e}{K_\mathrm{a}}\mathcal{F}_1 \tag{5}$$

and

$$\mathcal{F}_j = \sum_{i=0}^{K_\mathrm{a}-1} \alpha_i \sum_{n=1}^{\delta(K_\mathrm{t}-i)} \binom{\delta(K_\mathrm{t}-i)}{n} (-1)^n \sum_{b=0}^{(\nu-1)n} \beta_{n,b}^{(\nu)} \frac{(b+j)!}{n^{b+1+j}} \tag{6}$$

*Proof:* See Appendix B. ■

## Appendix A
## Proof of Lemma 1

The expectation $E[\mathcal{I}_{i,k}]$ is obtained after using (1) in [1, (11)], working the resulting integral out, and effecting some algebraic manipulations. Next, the key steps in the derivation are summarized. Substitution of (1) into [1, (11)] yields

$$E[\mathcal{I}_{i,k}] = \int_0^\infty \left(1 - \sum_{r=0}^{i} \alpha_{i,r} \frac{1}{\Gamma^{\delta(K_\mathrm{t}-i+r)}(\nu)} \gamma^{\delta(K_\mathrm{t}-i+r)}\left(\nu, \frac{2^\xi-1}{\rho_k}\right)\right) d\xi. \tag{7}$$

Using the finite sum expression for the incomplete gamma function and the binomial expansion, (7) can be rewritten as

$$E[\mathcal{I}_{i,k}] = \int_0^\infty d\xi - \sum_{r=0}^{i} \alpha_{i,r} \sum_{n=0}^{\delta(K_\mathrm{t}-i+r)} \binom{\delta(K_\mathrm{t}-i+r)}{n} (-1)^n \int_0^\infty e^{-n\frac{2^\xi-1}{\rho_k}} \left(\sum_{m=0}^{\nu-1} \frac{(2^\xi-1)^m}{\rho_k^m m!}\right)^n d\xi. \tag{8}$$

Noting that $\sum_{r=0}^{i} \alpha_{i,r} = 1$, the term corresponding to $n = 0$ in (8) cancels out with $\int_0^\infty d\xi$ and thus

$$E\left[\mathcal{I}_{i,k}\right] = -\sum_{r=0}^{i} \alpha_{i,r} \sum_{n=1}^{\delta(K_{\rm t}-i+r)} \binom{\delta(K_{\rm t}-i+r)}{n} (-1)^n \int_0^\infty e^{-n\frac{2^\xi-1}{\rho_k}} \left(\sum_{m=0}^{\nu-1} \frac{(2^\xi-1)^m}{\rho_k^m m!}\right)^n d\xi. \quad (9)$$

Applying the multinomial theorem to (9), we arrive at

$$E\left[\mathcal{I}_{i,k}\right] = -\sum_{r=0}^{i} \alpha_{i,r} \sum_{n=1}^{\delta(K_{\rm t}-i+r)} \binom{\delta(K_{\rm t}-i+r)}{n} (-1)^n \sum_{m_1+\ldots+m_\nu=n} \frac{n!}{m_1!\cdot\ldots\cdot m_\nu!} \frac{1}{c_m} \int_0^\infty e^{-n\frac{2^\xi-1}{\rho_k}} \frac{(2^\xi-1)^{b_m}}{\rho_k^{b_m}} d\xi \quad (10)$$

with $b_m = \sum_{r=1}^{\nu} (r-1)\, m_r$, and $c_m = \prod_{r=1}^{\nu} (r-1)!^{m_r}$.

With proper rearrangement of the multinomial terms, it can be shown that

$$\sum_{m_1+\ldots+m_\nu=n} \frac{n!}{m_1!\cdot\ldots\cdot m_\nu!} \frac{a^{b_m}}{c_m} = \sum_{b=0}^{(\nu-1)n} \beta_{n,b}^{(\nu)} a^b \quad (11)$$

with $\beta_{n,b}^{(\nu)}$ as defined in (3).

Plugging (11) into (10), we obtain

$$E\left[\mathcal{I}_{i,k}\right] = -\sum_{r=0}^{i} \alpha_{i,r} \sum_{n=1}^{\delta(K_{\rm t}-i+r)} \binom{\delta(K_{\rm t}-i+r)}{n} (-1)^n \sum_{b=0}^{(\nu-1)n} \beta_{n,b}^{(\nu)} \int_0^\infty e^{-n\frac{2^\xi-1}{\rho_k}} \frac{(2^\xi-1)^b}{\rho_k^b} d\xi. \quad (12)$$

The integrals in (12) are solved by making the change of variable $y = \frac{2^\xi-1}{\rho_k}$, using [2, 3.353.5] and the relation between the exponential integral and the incomplete gamma [2, 8.359], and some algebra, resulting in

$$\int_0^\infty e^{-n\frac{2^\xi-1}{\rho_k}} \frac{(2^\xi-1)^b}{\rho_k^b} d\xi = \log_2(e) e^{n/\rho_k} \frac{b!}{\rho_k^b} \Gamma\left(-b, \frac{n}{\rho_k}\right). \quad (13)$$

Substitution of (13) into (12) yields the final expression in [1, (14)].

## Appendix B

## Proof of Lemma 2

The low-SNR expansion follows from the expression of the ergodic sum-rate in [1, (18)], by replacing the products of exponentials and incomplete gamma functions with an approximate form. Based on [2, 8.357], it holds that

$$\frac{e^{n/\rho}}{\rho^b} \Gamma\left(-b, \frac{n}{\rho}\right) = \left(\frac{1}{n}\right)^{b+1} \frac{\rho}{b!} \sum_{m=0}^{R} \frac{(-1)^m (m+b)! \rho^m}{n^m} + o\left(\rho^{R+2}\right). \quad (14)$$

In order to approximate the ergodic sum-rate as $C = \dot{C}\left(0\right)\rho + \frac{\ddot{C}(0)}{2}\rho^2 + o\left(\rho^3\right)$, we use (14) with $R = 1$ in [1, (18)], which results in

$$C = \frac{-\log_2 e}{K_{\mathrm{a}}} \sum_{i=0}^{K_{\mathrm{a}}-1} \alpha_i \sum_{n=1}^{\delta(K_{\mathrm{t}}-i)} \binom{\delta(K_{\mathrm{t}}-i)}{n} (-1)^n \sum_{b=0}^{(\nu-1)n} \beta_{n,b}^{(\nu)} \left(\frac{b!\rho}{n^{b+1}} - \frac{(b+1)!\rho^2}{n^{b+2}}\right) + o\left(\rho^3\right). \quad (15)$$

Finally, $\dot{C}\left(0\right)$ and $\ddot{C}\left(0\right)$ are obtained after identifying (15) with $C = \dot{C}\left(0\right)\rho + \frac{\ddot{C}(0)}{2}\rho^2 + o\left(\rho^3\right)$.

## REFERENCES


[1] D. Morales-Jimenez and A. Lozano, “Ergodic sum-rate of proportional fair scheduling with multiple antennas,” submitted to IEEE International Symposium on Information Theory (ISIT 2013).

[2] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. San Diego: Academic Press, 2000.