On the Diagonal Distribution of a Complex Wishart Matrix and its Application to the Analysis of MIMO Systems

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***Abstract*—The statistical properties of Wishart matrices have been extensively used to analyze the performance of multiple-input multiple-output (MIMO) systems. In particular, the signal-to-noise ratio (SNR) output statistics of several MIMO systems depends on the diagonal distribution of a complex Wishart matrix. In this paper, we derive the joint density of the diagonal elements of a complex Wishart matrix, which follows a multivariate chi-square distribution. The density expression is in the form of an infinite series representation which converges rapidly and is easy to compute. This expression is used to obtain the distribution of the maximum of the diagonal elements, which allows analyzing the performance of two different MIMO systems under practical conditions. First, our statistical results are applied to the outage probability characterization of MIMO systems with receive antenna selection in spatially correlated Rayleigh fading. Then, the same results are used to analyze the outage probability of transmit beamforming systems under limited-rate feedback.**

***Index Terms*—Complex Wishart matrix, multivariate chi-square distribution, receive antenna selection, spatial correlation, transmit beamforming.**

I. Introduction

MULTIPLE antenna systems have been used for a long time to mitigate the effects of fading in wireless communications. More recently, multiple-input multiple-output (MIMO) systems have played an important role to satisfy the increasing demand for higher capacity and coverage [1–3]. Such systems may combine the use of space-time block codes (STBCs) at the transmitter [4] and receive diversity techniques such as antenna selection [5], which reduces complexity and the expensive RF chains at the receiver. When channel state information (CSI) is available at the transmitter, more sophisticated schemes can be employed to enhance the performance. The MIMO maximal ratio combining (MRC) system, also referred to as MIMO beamforming, relies on the joint MRC weights at both the transmitter and the receiver sides [6–11]. However, the performance of such systems is very often constrained by practical limitations as the antenna correlation, which reduces spatial diversity, or the limited-rate feedback, which has been dealt with codebook-based beamforming approaches.

The statistical properties of Wishart matrices have been widely used to analyze the performance of MIMO systems [12–16]. Particularly, the diagonal entries of complex Wishart matrices are employed for characterizing the signal-to-noise ratio (SNR) statistics at the output of certain MIMO systems under practical limitations. The distribution of the maximum of the diagonal elements can be used to characterize the output SNR of diversity receivers (MIMO with receive antenna selection) under spatially correlated fading (see, e.g., [5, 17, 18]). Also, the use of the diagonal distribution of complex Wishart matrices has been pointed out in [19] as an approach to the performance analysis of beamforming systems with limited feedback (codebook-based). The derivation of tractable analytical expressions for the SNR statistics of such systems is very important in order to evaluate performance measures such as the outage probability, bit error rate (BER), or system capacity. The reasons above motivate us to focus on the diagonal distribution of a complex Wishart matrix, which is a particular multivariate chi-square distribution derived from complex Gaussian variables.

In the *area of multivariate analysis*, there is a rich body of works considering the joint distribution of the diagonal elements of real Wishart matrices; equivalently, multivariate chi-square distributions derived from real Gaussian random variables [20–24]. Whereas the characteristic function (CF) is well known [22], the joint probability density function (PDF) is rather more complicated. Different approaches to the joint PDF have been proposed in the literature. An infinite series expansion for the density in terms of Laguerre polynomials was first given in [22]. Following the same approach, later work by Royen [23] provided new Laguerre expansions with improved convergence. In [24], Miller *et al.* derived expansions for the PDF in terms of Bessel functions for the bivariate and trivariate cases. However, the case of underlying complex Gaussian random variables, i.e., complex Wishart matrices, has not been sufficiently investigated. Only very recently, Hagedorn *et al.* [25] have derived corresponding expansions for the trivariate case, extending previous results in [24] to the complex case. To the best of the authors' knowledge, the case of k-variate chi-square ($k > 3$) from a complex

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Wishart matrix with arbitrary correlation is not available in the literature. Moreover, the expansions in [25] for $k = 3$ are given via a product of Bessel functions, which does not lend itself into manipulations in the context of performance analysis of MIMO systems.

In the *communications theory context*, the available statistical results on the multivariate chi-square distribution have been applied to the performance analysis of MIMO systems [5, 17–19]. However, the analysis is often limited due to the lack of results on the diagonal distribution of complex Wishart matrices. Moreover, analytical closed-form expressions (e.g., BER) are rarely provided due to the awkward form of the joint PDF. In [18], the derived BER expressions for multi-branch selection combining (SC) over spatially correlated fading are in the form of a multiple integral involving the joint CF. Also, BER results in [19] for codebook-based transmit beamforming are again given in a multiple integral form. In some other works, the analysis is carried out under certain assumptions such as a real channel correlation, which leads to the statistics of a real Wishart matrix. In [17], closed-form BER results are provided for dual-branch selection diversity assuming real correlation among branches. Also, the real exponential correlation model is assumed in [5] for more than 3 antennas to analyze the performance of MIMO systems with SC.

In this paper, we derive a new expression for the joint cumulative distribution function (CDF) and joint PDF of a k-variate chi-square distribution from a complex Wishart matrix with *arbitrary* correlation. The main contributions of this paper can be summarized as follows:

- An infinite series representation is given for the joint CDF and PDF of the multivariate chi-square distribution from a complex Wishart matrix, based on previous results for the real case in [23]. These expressions are straightforward for numerical work and the expansions converge rapidly. Further performance analysis of MIMO systems such as exact BER analysis is made possible due to the series expansions in terms of Laguerre polynomials.
- The derived expression for the joint CDF is used to obtain a new series expansion for the distribution of the maximum of k correlated chi-square random variables.
- Although seemingly complex, the computation of this series is mathematically tractable. An efficient MATHEMATICA™ algorithm is provided for rapid computation of the coefficients.
- The newly derived series representation is applied to the analysis of MIMO communication systems under practical conditions. First, our statistical results are applied to the outage probability analysis of MIMO systems with SC in arbitrarily correlated Rayleigh fading channels. This analysis extends the results in [5] to any number of antennas with arbitrary correlation. Moreover, our analysis is applicable to transmit antenna selection systems, which are especially interesting in the uplink direction in order to reduce the number of RF chains at the terminal side. Our results are in the form of a single series expansion in terms of the Laguerre polynomials, which facilitates the computation and makes further closed-form analysis (e.g. BER analysis) possible. Finally, the same statistical results are applied to obtain a new expression for the outage probability of codebook-based transmit beamforming systems with MRC at the receiver side.

The rest of this paper is organized as follows. The joint CDF and PDF of the diagonal elements of a complex Wishart matrix are derived in Section II. Based on the results, the distribution of the maximum of the diagonal elements is discussed in Section III. Then, these statistical results are applied to the performance analysis of MIMO systems with SC under arbitrarily correlated fading and then with codebook-based beamforming in Section IV. Section V provides some numerical results and we conclude the paper in Section VI.

Throughout the paper, the following vector and matrix notations are used: bold lower-case for vectors, bold upper-case for matrices, superscripts T and H for the transpose and the Hermitian transpose, respectively, $\|\cdot\|$ for the Euclidean vector norm, $|\cdot|$ for the matrix determinant, $\mathsf{diag}(\cdot)$ for the diagonal elements of a matrix, and $\mathsf{Diag}\,(a_1, \ldots, a_k)$ for the diagonal matrix with diagonal elements $(a_1, \ldots, a_k)$. Also, we use $\mathtt{Re}\{\cdot\}$ and $\mathtt{E}[\cdot]$ for the real part and the expectation operators, respectively.

II. The Diagonal Distribution of a Complex Wishart Matrix

A. Preliminaries

Let $\mathbf{X}_j = [X_{1,j}, X_{2,j}, \ldots, X_{k,j}]^T$ be the j-th sample of a k-dimensional zero-mean complex Gaussian process ($j = 1, 2, \ldots, p$), where $\{\mathbf{X}_j\}$ are mutually independent and identically distributed (i.i.d.). The covariance matrix for each Gaussian random vector is $\mathbf{R}$, i.e., $\mathbf{X}_j \sim \mathcal{CN}(\mathbf{0}, \mathbf{R})$. Then, the matrix $\mathbf{S} = \sum_{j=1}^{p} \mathbf{X}_j \mathbf{X}_j^H$ has a complex Wishart distribution denoted by $\mathcal{CW}_k(p, \mathbf{R})$, and the diagonal elements of $\mathbf{S}$, defined by

$$\mathsf{diag}(\mathbf{S}) = (Y_1, Y_2, \ldots, Y_k) = \left(\sum_{j=1}^{p} |X_{1,j}|^2, \sum_{j=1}^{p} |X_{2,j}|^2, \ldots, \sum_{j=1}^{p} |X_{k,j}|^2 \right), \tag{1}$$

are chi-square distributed with $2p$ degrees of freedom, i.e., $Y_i \sim \chi^2(0, 2p)$ having PDF

$$f_{Y_i}(y) = \frac{1}{2^p \Gamma(p)} y^{p-1} e^{-\frac{y}{2}}, \quad (i = 1, 2, \ldots, k). \tag{2}$$

Note that this is just a scaled version of the gamma distribution, $f_{Y_i}(y) = \frac{1}{2} g_p\left(y/2\right)$, with the gamma density defined as

$$g_\alpha(x) = \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)}. \tag{3}$$

The joint distribution of $(Y_1, Y_2, \ldots, Y_k)$, i.e., the diagonal distribution of the complex Wishart matrix $\mathbf{S}$, is a k-variate central chi-square distribution with correlation structure induced by $\mathbf{R}$. The rest of this section is devoted to the statistical analysis of this distribution. Specifically, exact infinite series expansions are provided for both the joint CDF and joint PDF denoted by $F_{Y_1,Y_2,\ldots,Y_k}(y_1, y_2, \ldots, y_k)$ and $f_{Y_1,Y_2,\ldots,Y_k}(y_1, y_2, \ldots, y_k)$, respectively.

B. Joint CDF

Our objective is to find a tractable and easily computable expansion for the CDF by following a similar approach to that in [23], where results are provided for the real case. In the complex case, the correlation structure of the underlying Gaussian random variables is different, leading to a slightly different CF with a complex correlation matrix. In the extension to the complex case, we basically apply the approach in [23] to the CF, and then the binomial expansion and Fourier inversion are used to obtain the series. As a different CF with a complex correlation matrix is involved, the validity of the derivations in [23] has been revised and carefully checked. Then, our analysis diverges from that in [23] by using a more convenient representation and rearranging the series when it comes to the distribution of the maximum of the diagonal elements (see section III).

The starting point is the CF for the diagonal distribution of a real Wishart matrix, which is well known [22, 23]. After an extension to the complex case, it is possible to write the CF of the diagonal elements of a complex Wishart as

$$\Phi(t_1,\ldots,t_k) = \mathtt{E}\left[e^{i(t_1Y_1+\cdots+t_kY_k)}\right] = |\mathbf{I} - i\mathbf{RT}|^{-p}, \quad (4)$$

where $\mathbf{I}$ is the k-dimensional identity matrix, $\mathbf{R} = \mathtt{E}\left[\mathbf{X}_j\mathbf{X}_j^H\right]$ is the covariance or also referred to as correlation matrix, and $\mathbf{T} = \mathsf{Diag}(t_1,\ldots,t_k)$.

Using the approach in [23] to (4), we arrive at the following representation of the CF:

$$\Phi(t_1,\ldots,t_k) = |\mathbf{I} - (\mathbf{I} - \mathbf{WRW})\mathbf{U}|^{-p}\prod_{j=1}^{k}(1-u_j)^p, \quad (5)$$

where $\mathbf{W} = \mathsf{Diag}(w_1,\ldots,w_k)$, with w_j any scale factors, and $\mathbf{U} = \mathsf{Diag}(u_1,\ldots,u_k)$, with

$$u_j = 1 - \left(1 - i\frac{t_j}{w_j^2}\right)^{-1}. \quad (6)$$

Note that the Fourier transform of $w \cdot g_{p+n}^{(n)}(w\cdot y_j)$, where $g_{p+n}^{(n)}(x)$ denotes the n-th derivative of the gamma density $g_{p+n}(x)$, is given by

$$u_j^n(1-u_j)^p, \quad \text{for } \mathtt{Re}\left\{(1-u_j)^{\frac{1}{2}}\right\} > 0. \quad (7)$$

For convenience, we define $g_p^{(-1)}(x)$ as the gamma CDF, i.e.,

$$g_p^{(-1)}(x) \triangleq G_p(x) = \frac{\gamma(p,x)}{\Gamma(p)} = 1 - e^{-x}\sum_{j=0}^{p-1}\frac{x^j}{j!}. \quad (8)$$

The CF in (5) can be expressed as an infinite series by replacing $|\mathbf{I} - (\mathbf{I} - \mathbf{WRW})\mathbf{U}|^{-p}$ with its binomial expansion and, then, the joint CDF is obtained by the Fourier inversion of the series. Thus, after considering (7) as well as previous definition in (8), the following expansion for the joint CDF is obtained:

$$F_{Y_1,Y_2,\ldots,Y_k}(y_1,y_2,\ldots,y_k) = \sum_{n=0}^{\infty}\sum_{(n=n_1+\cdots+n_k)} c(n_1,\ldots,n_k)\prod_{j=1}^{k} g_{p+n_j}^{(n_j-1)}\left(w_j^2 y_j\right), \quad (9)$$

where $\sum_{(n=n_1+\cdots+n_k)}$ denotes the summation over all possible integer partitions satisfying $n = n_1 + \cdots + n_k$, and the coefficients $c(n_1,\ldots,n_k)$ depend on p, the scale factors w_j, and the complex correlation matrix $\mathbf{R}$. The scale factors are chosen to assure the convergence of (9), which is guaranteed under the condition [1]

$$\|\mathbf{I} - \mathbf{WRW}\|_2 < 1, \quad (10)$$

where $\|\mathbf{A}\|_2$ denotes the spectral norm of $\mathbf{A}$, i.e. the square root of the maximum eigenvalue of $\mathbf{A}^H\mathbf{A}$.

Now, using the Rodrigues' formula [26, Eq. (22.11.6)], we can write

$$g_{p+n}^{(n-1)}(x) = \frac{(n-1)!}{(p+n-1)!}e^{-x}x^pL_{n-1}^p(x), \quad (11)$$

where $L_n^a(x)$ is the n-th order generalized Laguerre polynomial, given by

$$L_n^a(x) = \sum_{i=0}^{n}(-1)^i\binom{n+a}{n-i}\frac{x^i}{i!}. \quad (12)$$

Thus, the joint CDF can be rewritten in terms of the well-known Laguerre polynomials as

$$F_{Y_1,Y_2,\ldots,Y_k}(y_1,y_2,\ldots,y_k) = \sum_{n=0}^{\infty}\sum_{(n=n_1+\cdots+n_k)} c(n_1,\ldots,n_k)\prod_{j=1}^{k}\Delta_{n_j}^p\left(w_j^2y_j\right), \quad (13)$$

with

$$\Delta_n^p(x) \triangleq \begin{cases} G_p(x) = 1 - e^{-x}\sum_{j=0}^{p-1}\frac{x^j}{j!}, & n = 0,\\ g_{p+n}^{(n-1)}(x) = \frac{(n-1)!}{(p+n-1)!}e^{-x}x^pL_{n-1}^p(x), & n > 0. \end{cases} \quad (14)$$

What remains to complete the expansion of the joint CDF in (13) is to find an easily computable expression for the coefficients $c(n_1,\ldots,n_k)$, which is addressed by the following proposition.

Proposition 1: The coefficients $c(n_1,\ldots,n_k)$ for the series expansion in (13) can be obtained as the coefficients of the n-order homogeneous polynomial θ_n as

$$\theta_n(u_1,u_2,\ldots,u_k) = \sum_{(n=n_1+\cdots+n_k)} c(n_1,\ldots,n_k)\prod_{j=1}^{k}u_j^{n_j}$$
$$= \sum_{(n=\ell_1+2\ell_2+\cdots+k\ell_k)}\frac{\Gamma(p+\ell_1+\cdots+\ell_k)}{\Gamma(p)}\prod_{j=1}^{k}\frac{(-D_j)^{\ell_j}}{\ell_j!}, \quad (15)$$

where D_j denotes the polynomial generated from the determinants of the submatrices of $\mathbf{A} = \mathbf{I} - \mathbf{WRW}$ as

$$D_j = (-1)^j\sum_{\mathsf{size}(\mathcal{S})=j}|\mathbf{A}_{\mathcal{S}}|\prod_{m\in\mathcal{S}}u_m, \quad (16)$$

with $\mathbf{A}_{\mathcal{S}}$ representing the submatrix of $\mathbf{A}$ with the rows and columns specified by the non-empty subset $\mathcal{S} \subseteq \{1,2,\ldots,k\}$, and $\sum_{\mathsf{size}(\mathcal{S})=j}$ denoting the summation computed over all possible subsets $\mathcal{S}$ whose size is j.

Proof: See Appendix A. ■

[1]The sufficient condition for convergence of the series follows from the binomial expansion of (5), and a proof can be obtained from [23, Theorem 2.1].

It should be noted that all the coefficients in the series for a given value of n are obtained from the polynomial θ_n, which makes the computation very efficient. Also, it is likely to find a great number of null coefficients, especially for a large number of variables (k). As a consequence, the terms in the series to be computed are reduced to those appearing in the polynomial θ_n, which significantly decreases the computation cost. Appendix B includes a MATHEMATICA™ program with an efficient algorithm to compute the polynomial θ_n, where the coefficients $c(n_1,\ldots,n_k)$ can be easily extracted from.

C. Joint PDF

The joint PDF of the diagonal elements of a complex Wishart matrix $f_{Y_1,Y_2,\ldots,Y_k}(y_1,y_2,\ldots,y_k)$ can be obtained by differentiation of the CDF in (13), which allows us to write

$$f_{Y_1,Y_2,\ldots,Y_k}(y_1,y_2,\ldots,y_k) = \sum_{n=0}^{\infty} \sum_{(n=n_1+\cdots+n_k)} c(n_1,\ldots,n_k) \prod_{j=1}^{k} \frac{d\left[\Delta_{n_j}^{p}\left(w_j^2 y_j\right)\right]}{dy_j}. \quad (17)$$

The derivative of the delta function, defined in (14), is given by

$$\frac{d\Delta_n^p(x)}{dx} = g_{p+n}^{(n)}(x) = \frac{n!}{(p+n-1)!} e^{-x} x^{p-1} L_n^{p-1}(x). \quad (18)$$

Then, by substituting (18) into (17), the expansion for the joint PDF is expressed as

$$f_{Y_1,Y_2,\ldots,Y_k}(y_1,y_2,\ldots,y_k) = \sum_{n=0}^{\infty} \sum_{(n=n_1+\cdots+n_k)} c(n_1,\ldots,n_k) \prod_{j=1}^{k} w_j^2 g_{p+n_j}^{(n_j)}\left(w_j^2 y_j\right). \quad (19)$$

Note that the joint PDF is given in terms of the n-th derivative of the gamma density, which in turn is directly related to the generalized Laguerre polynomials, as shown in (18).

III. Distribution of the Maximum of the Diagonal Elements of a Complex Wishart Matrix

The diagonal of a complex Wishart matrix is a type of multivariate central chi-square distribution with a certain underlying complex correlation matrix. The distribution of the maximum of these correlated variables is of special interest within the performance analysis of many communication systems [5, 17–19, 27]. This section presents expressions for the CDF and PDF of this distribution.

A. CDF and PDF

Let us consider Z to be the maximum of the k correlated central chi-square random variables, i.e., $Z = \max\{Y_1,Y_2,\ldots,Y_k\}$. On the one hand, the CDF of Z is given by

$$\begin{aligned} F_Z(z) &= \Pr\{Z \le z\} \\ &= \Pr\{Y_1 \le z, Y_2 \le z, \ldots, Y_k \le z\} \\ &= F_{Y_1,Y_2,\ldots,Y_k}(z,z,\ldots,z). \end{aligned} \quad (20)$$

That is, the CDF of the maximum is obtained by setting the argument of the joint CDF in (13) to $(z,z,\ldots,z)$, which yields

$$F_Z(z) = \sum_{n=0}^{\infty} \sum_{(n=n_1+\cdots+n_k)} c(n_1,\ldots,n_k) \prod_{j=1}^{k} \Delta_{n_j}^{p}\left(w_j^2 z\right), \quad (21)$$

where $\Delta_n^p(x)$ are as previously defined in (14), and $c(n_1,\ldots,n_k)$ are obtained as in Proposition 1. On the other hand, the PDF of Z can be derived by differentiation of (21), thus giving

$$f_Z(z) = \sum_{n=0}^{\infty} \sum_{(n=n_1+\cdots+n_k)} c(n_1,\ldots,n_k) \sum_{j=1}^{k} w_j^2 g_{p+n_j}^{(n_j)}\left(w_j^2 z\right) \prod_{\substack{i\in\{1,\ldots,k\} \\ i\neq j}} \Delta_{n_i}^{p}\left(w_i^2 z\right), \quad (22)$$

with $g_{p+n}^{(n)}(x)$ as previously defined in (18).

B. Series rearrangement

The derived series expansions for both the CDF and PDF of Z can be rearranged under the assumption of having a single scale factor $w = w_1 = w_2 = \cdots = w_k$. Let $P^{(n)} = \left\{P_i^{(n)}\right\}$, for $i = 1,\ldots,s_n$, be the set of all integer partitions of n into k elements and s_n its size. Then, there is one term (coefficient) in the series for each ordered sequence $(n_1,\ldots,n_k)$ that satisfies $n = n_1+\cdots+n_k$. Note that the sequence $(n_1,\ldots,n_k)$ is just some permutation of $P_i^{(n)} = \left\{p_{i,1}^{(n)},\ldots,p_{i,k}^{(n)}\right\}$. Under the single scale factor assumption, the CDF series becomes

$$F_Z(z) = \sum_{n=0}^{\infty} \sum_{(n=n_1+\cdots+n_k)} c(n_1,\ldots,n_k) \prod_{j=1}^{k} \Delta_{n_j}^{p}\left(w^2 z\right), \quad (23)$$

where the dependence of the product with the ordered sequence $(n_1,\ldots,n_k)$ vanishes since all the delta functions have the same argument. Now, the product in (23) depends only on the scale factor w and on the integer partition $P_i^{(n)}$. Then, all the terms corresponding to permutations of the same integer partition can be grouped under a new term with coefficient $\hat{c}\left(P_i^{(n)}\right)$, thus yielding the rearranged series

$$F_Z(z) = \sum_{n=0}^{\infty} \sum_{i=1}^{s_n} \hat{c}\left(p_{i,1}^{(n)},\ldots,p_{i,k}^{(n)}\right) \prod_{j=1}^{k} \Delta_{p_{i,j}^{(n)}}^{p}\left(w^2 z\right), \quad (24)$$

where $\hat{c}\left(p_{i,1}^{(n)},\ldots,p_{i,k}^{(n)}\right) = \sum_{(n_1,\ldots,n_k)\in P_i^{(n)*}} c(n_1,\ldots,n_k)$ with $P_i^{(n)*}$ being the set of all possible permutations of $P_i^{(n)}$. It is emphasized that, with this rearrangement, the number of terms in the series for a given n is reduced by the number of all possible permutations of each integer partition of n, which allows for an easy and rapid computation. In Appendix B, a MATHEMATICA™ program is provided to efficiently compute $\hat{c}\left(p_{i,1}^{(n)},\ldots,p_{i,k}^{(n)}\right)$ for $i = 1,\ldots,s_n$ and $n = 0,\ldots,N_{max}$, where N_{max} is the truncation limit. As it will be shown in the numerical section, the time to compute all the coefficients for a typical truncation limit $N_{max} = 8$, which gives an accurate CDF representation (4-figure accuracy), is less than one second.

C. Special cases and approximations

1) The bivariate case ($k = 2$): The joint CDF in this case is given by

$$F_{Y_1,Y_2}(y_1, y_2) = \sum_{n=0}^{\infty} \sum_{(n=n_1+n_2)} c(n_1, n_2) \Delta_{n_1}^p (w_1^2 y_1) \Delta_{n_2}^p (w_2^2 y_2), \tag{25}$$

with $\Delta_n^p(x)$ as previously defined.

Under the single scale factor assumption $w_1 = w_2 = 1$, the polynomial in (15) to obtain the coefficients $c(n_1, n_2)$ reduces to

$$\theta_n(u_1, u_2) = \begin{cases} 0, & n \quad \text{odd}, \\ \frac{(p-1+n/2)!}{(p-1)!(n/2)!} |r|^n (u_1 u_2)^{n/2}, & n \quad \text{even}, \end{cases} \tag{26}$$

where r is the complex value from the correlation matrix $\mathbf{R} = \begin{pmatrix} 1 & r \\ r^* & 1 \end{pmatrix}$. Thus, there is only one non-null coefficient for each even value of n given by

$$c(n/2, n/2) = \frac{(p-1+n/2)!}{(p-1)!(n/2)!} |r|^n, \tag{27}$$

and the series for the joint CDF in the bivariate case can be rewritten as

$$F_{Y_1,Y_2}(y_1, y_2) = G_p(y_1) G_p(y_2) + \frac{e^{-y_1} e^{-y_2} y_1^p y_2^p}{(p-1)!} \sum_{n=1}^{\infty} \frac{(n-1)!}{n(p+n-1)!} |r|^{2n} L_{n-1}^p(y_1) L_{n-1}^p(y_2), \tag{28}$$

where $G_p(x)$ is the gamma CDF, defined in (8). Finally, the distribution of the maximum is obtained by setting $y_1 = y_2 = z$ in (28), which yields

$$F_Z(z) = G_p(z)^2 + \frac{e^{-2z} z^{2p}}{(p-1)!} \sum_{n=1}^{\infty} \frac{(n-1)!}{n(p+n-1)!} |r|^{2n} L_{n-1}^p(z)^2. \tag{29}$$

2) Approximation for $z \to 0$: In order to obtain an approximation of $F_Z(z)$ for small values of z, we first study the behaviour of the delta functions $\Delta_n^p(x)$ in the limit $x \to 0$. After substituting the exponential function by its Taylor expansion in (14) and some straightforward algebra, we arrive at

$$\lim_{x \to 0} \Delta_n^p(x) = \frac{x^p}{p!}. \tag{30}$$

Then, (30) is used in (21) to obtain the following approximation which holds for $z \to 0$

$$F_Z(z) \approx \alpha \left(\frac{1}{p!}\right)^k z^{pk} \prod_{j=1}^{k} w_j^{2p}, \tag{31}$$

where α is the sum of all the coefficients, namely

$$\alpha = \sum_{n=0}^{\infty} \sum_{(n=n_1+\ldots+n_k)} c(n_1, \ldots, n_k). \tag{32}$$

Also, it is possible to see from (47) that the sum of coefficients can be obtained as

$$\alpha = |\mathbf{I} - (\mathbf{I} - \mathbf{WRW})|^{-p} = |\mathbf{WRW}|^{-p}. \tag{33}$$

Then, the approximation for the CDF of the maximum is rewritten as

$$F_Z(z) \approx |\mathbf{WRW}|^{-p} \left(\frac{1}{p!}\right)^k z^{pk} \prod_{j=1}^{k} w_j^{2p}, \tag{34}$$

which, under the single scale factor assumption, simplifies to

$$F_Z(z) \approx |\mathbf{R}|^{-p} \left(\frac{1}{p!}\right)^k z^{pk}. \tag{35}$$

IV. Applications

In this section, the derived expressions for the CDF of the maximum of the diagonal elements of a complex Wishart matrix are applied to the performance analysis of two different MIMO systems. First, the outage probability is analyzed for MIMO systems with receive antenna selection under arbitrarily correlated fading. Then, the same analytical approach is applied to codebook based transmit beamforming systems with MRC at the receiver side.

A. Receive antenna selection in MIMO spatially correlated fading

Consider a MIMO communication system with N_T transmit and N_R receive antennas, where all antennas are used for transmission and only a single receive antenna, which maximizes the instantaneous SNR, is selected. The MIMO fading channel is modeled by the $N_R \times N_T$ random matrix $\mathbf{H}$, defined according to the well-known Kronecker spatial correlation model [28]

$$\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{N_T}] = \mathbf{R}_R^{\frac{1}{2}} \mathbf{G} \mathbf{R}_T^{\frac{1}{2}}, \tag{36}$$

where $\mathbf{R}_T$ and $\mathbf{R}_R$ are the transmit and receive correlation matrices, respectively, whereas the entries of $\mathbf{G}$ are i.i.d. complex Gaussian random variables with zero mean and unit variance. Transmit antennas are assumed to be spaced far enough so that the transmitter side correlation is negligible, or $\mathbf{R}_T = \mathbf{I}$, and thus, spatial correlation only appears at the receiver side. In this case, the column vectors of the channel matrix $\mathbf{h}_j = [h_{1j}, h_{2j}, \ldots, h_{N_R j}]^T$, for $j = 1, \ldots, N_T$, are zero-mean i.i.d. complex Gaussian processes with covariance matrix $\mathbf{R}_R$, i.e., $\mathbf{h}_j \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_R)$. Also, note that the matrix $\mathbf{S}_h = \sum_{j=1}^{N_T} \mathbf{h}_j \mathbf{h}_j^H$ has the complex Wishart distribution $\mathcal{CW}_{N_R}(N_T, \mathbf{R}_R)$ and its N_R diagonal elements are the square norms of the row vectors of $\mathbf{H}$.

The baseband complex envelope of the received signal after the matched filter is expressed as $\mathbf{y} = \mathbf{Hx} + \mathbf{n}$, where $\mathbf{n}$ is the N_R-dimensional white noise vector whose elements are complex Gaussian random variables with zero mean and variance σ_n^2. The transmitted signal is denoted by the column vector $\mathbf{x}$ and the total average transmit power is normalized to one, i.e., $\mathrm{E}[\mathbf{x}^H \mathbf{x}] = 1$, and evenly distributed among all the antennas. Under these assumptions, the average SNR at each receiver branch is given by $\bar{\gamma} = \frac{1}{\sigma_n^2}$.

The receive antenna selection technique, also referred to as SC, selects the antenna which maximizes the instantaneous SNR [5]. In the considered MIMO system with SC, we also assume that CSI is perfectly known at the receiver but not available at the transmitter, where some diversity technique is applied (e.g., orthogonal STBCs). Then, the instantaneous SNR at the output of the combiner is given by

$$\gamma = \frac{\bar{\gamma}}{N_T} Z, \tag{37}$$

where

$$Z = \max_{1 \leq i \leq N_R} \sum_{j=1}^{N_T} |h_{ij}|^2 \tag{38}$$

is the maximum of the squared norms of the row vectors of the channel matrix, which are the diagonal elements of the complex Wishart matrix $\mathbf{S}_h$. Therefore, the CDF of Z can be obtained from the derived series expansion in (24) as

$$F_Z(z) = \sum_{n=0}^{\infty} \sum_{i=1}^{s_n} \hat{c}\left(p_{i,1}^{(n)}, \ldots, p_{i,N_R}^{(n)}\right) \prod_{j=1}^{N_R} \Delta_{p_{i,j}^{(n)}}^{N_T}\left(w^2 z\right), \tag{39}$$

where the coefficients $\hat{c}\left(p_{i,1}^{(n)}, \ldots, p_{i,N_R}^{(n)}\right)$ depend on N_T, w, and the correlation matrix $\mathbf{R}_R$, and can be easily computed with the algorithm provided in Appendix B. Recall that the delta functions, as defined in (14), are basically a scaled version of the generalized Laguerre polynomials multiplied by the exponential function.

As a result, the outage probability for the MIMO system with SC under spatially correlated fading is given by

$$\begin{aligned} P_{out}(x) &\triangleq \Pr\{\gamma \leq \gamma_0\} = \Pr\left\{Z \leq N_T \frac{\gamma_0}{\bar{\gamma}}\right\} = F_Z\left(N_T \frac{1}{x}\right) \\ &= \sum_{n=0}^{\infty} \sum_{i=1}^{s_n} \hat{c}\left(p_{i,1}^{(n)}, \ldots, p_{i,N_R}^{(n)}\right) \prod_{j=1}^{N_R} \Delta_{p_{i,j}^{(n)}}^{N_T}\left(\frac{w^2 N_T}{x}\right), \end{aligned} \tag{40}$$

where γ_0 is the outage threshold and $x \triangleq \frac{\bar{\gamma}}{\gamma_0}$ is the normalized average SNR. To the best of the authors' knowledge, this expression for the outage probability is original and novel. Also, it is emphasized that (40) is a general expression valid for an *arbitrary* correlation matrix and *any* number of receive antennas. Recently in [5], the same system was analyzed but only under the assumption of a real exponential correlation matrix for $N_R > 3$ due to the lack of results for the diagonal distribution of a complex Wishart matrix. This was however a valid correlation model only when the antennas are placed in a uniform linear array [29] and therefore, our analysis extends the results in [5] to the general arbitrary correlation case.

1) Diversity Order: After taking (35) into account, the outage probability in the high SNR regime can be approximated by

$$P_{out}(x) \approx |\mathbf{R}_R|^{-N_T} \left(\frac{1}{N_T!}\right)^{N_R} \frac{N_T^{N_T N_R}}{x^{N_T N_R}}, \tag{41}$$

which suggests that a diversity order of $N_T N_R$ is achieved. The same result has been obtained in [5], thus confirming the validity of our derivations.

2) Dual branch case ($N_R = 2$): For a dual branch receiver, the simplified expression for the bivariate case in (29) can be used to arrive at the outage probability, which is given by

$$\begin{aligned} P_{out}(x) &= G_{N_T}\left(\frac{N_T}{x}\right)^2 + \\ &\frac{e^{\frac{-2N_T}{x}}}{x^{2N_T}} \frac{N_T^{2N_T}}{(N_T-1)!} \sum_{n=1}^{\infty} \frac{(n-1)!}{n\,(N_T+n-1)!} |r|^{2n} L_{n-1}^{N_T}\left(\frac{N_T}{x}\right)^2, \end{aligned} \tag{42}$$

where r is the out-diagonal element of the correlation matrix $\mathbf{R}_R$. As shown by the simplified expression, the outage probability increases with $|r|$, which can be dealt by increasing the number of transmit antennas N_T.

B. Transmit beamforming under limited-rate feedback

Here, we consider a MIMO MRC system with N_T transmit and N_R receive antennas, where codebook-based transmit beamforming is applied together with MRC at the receiver. The MIMO channel is modeled by the $N_R \times N_T$ random matrix $\mathbf{H}$, whose entries are i.i.d. complex Gaussian random variables with zero mean and unit variance. The beamformer codebook matrix $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_L]$ consists of L different $N_T \times 1$ column vectors $\mathbf{b}_i$, for $i = 1, \ldots, L$. For each channel realization, the receiver selects a beamformer vector which maximizes the instantaneous SNR and reports it back to the transmitter over a finite-rate feedback channel using $\log_2(L)$ bits. In this case, the complex envelope of the received signal after the matched filter (before the MRC processing) is expressed as $\mathbf{y} = \mathbf{H}\hat{\mathbf{b}}x + \mathbf{n}$, where x is the transmitted symbol, $\hat{\mathbf{b}}$ is the selected beamformer vector that satisfies

$$\hat{\mathbf{b}} = \arg\max_{\mathbf{b} \in \mathbf{B}} \|\mathbf{H}\mathbf{b}\|^2, \tag{43}$$

and $\mathbf{n}$ is the N_R-dimensional white noise vector whose elements are complex Gaussian random variables with zero mean and variance σ_n^2. The total average transmit power is normalized to one, and consequently, $\mathrm{E}\left[|x|^2\right] = 1$ and $\|\mathbf{b}_i\|^2 = 1$, for $i = 1, \ldots, L$. Thus, the average SNR at each receive antenna is given by $\bar{\gamma} = \frac{1}{\sigma_n^2}$. Then, assuming the channel matrix $\mathbf{H}$ perfectly known at the receiver, the instantaneous SNR after the MRC processing is expressed as $\gamma = \bar{\gamma} Z$ with

$$Z = \max_{1 \leq i \leq L} \|\mathbf{H}\mathbf{b}_i\|^2 = \max_{1 \leq i \leq L} \sum_{j=1}^{N_R} |\mathbf{h}_j \mathbf{b}_i|^2, \tag{44}$$

where $\{\mathbf{h}_j\}$ are the row vectors of $\mathbf{H}$. Once again, Z is the maximum of the squared norms $v_i = \|\mathbf{H}\mathbf{b}_i\|^2$, for $i = 1, \ldots, L$, which can be identified as the diagonal elements of a certain complex Wishart matrix. Specifically, v_i are the diagonal elements of the matrix $\mathbf{S}_{bf} = \sum_{j=1}^{N_R} \mathbf{k}_j \mathbf{k}_j^H$, with $\mathbf{k}_j = \mathbf{B}^H \mathbf{h}_j^H$, which has the complex Wishart distribution $\mathcal{CW}_L(N_R, \mathbf{B}^H \mathbf{B})$. Equivalently, Z is the maximum of L correlated central chi-square variables with $2N_R$ degrees of freedom and underlying correlation matrix $\mathbf{R}_{bf} = \mathbf{B}^H \mathbf{B}$, i.e., with correlation determined by the codebook matrix. Hence, the CDF of Z can be expressed by the series expansion in (24) as

$$F_Z(z) = \sum_{n=0}^{\infty} \sum_{i=1}^{s_n} \hat{c}\left(p_{i,1}^{(n)}, \ldots, p_{i,L}^{(n)}\right) \prod_{j=1}^{L} \Delta_{p_{i,j}^{(n)}}^{N_R}\left(w^2 z\right), \tag{45}$$

TABLE I
CORRELATION MATRICES CONSIDERED IN THE PERFORMANCE ANALYSIS OF THE SC SYSTEM.

$\mathbf{R}_1$	$\begin{pmatrix} 1 & 0.50 & 0.25+0.33j & -0.10+0.33j \\ 0.50 & 1 & 0.17-0.33j & -0.17 \\ 0.25-0.33j & 0.17+0.33j & 1 & 0.33-0.25j \\ -0.10-0.33j & -0.17 & 0.33+0.25j & 1 \end{pmatrix}$
$\mathbf{R}_2$	$\begin{pmatrix} 1 & 0.50 & 0.25+0.33j & -0.10+0.33j & 0.12j & 0.11+0.17j \\ 0.50 & 1 & 0.17-0.33j & -0.17 & 0.17-0.17j & -0.17 \\ 0.25-0.33j & 0.17+0.33j & 1 & 0.33-0.25j & -0.33 & 0.33-0.17j \\ -0.10-0.33j & -0.17 & 0.33+0.25j & 1 & 0.33-0.11j & 0.12 \\ -0.12j & 0.17+0.17j & -0.33 & 0.33+0.11j & 1 & -0.20 \\ 0.11-0.17j & -0.17 & 0.33+0.17j & 0.12 & -0.20 & 1 \end{pmatrix}$

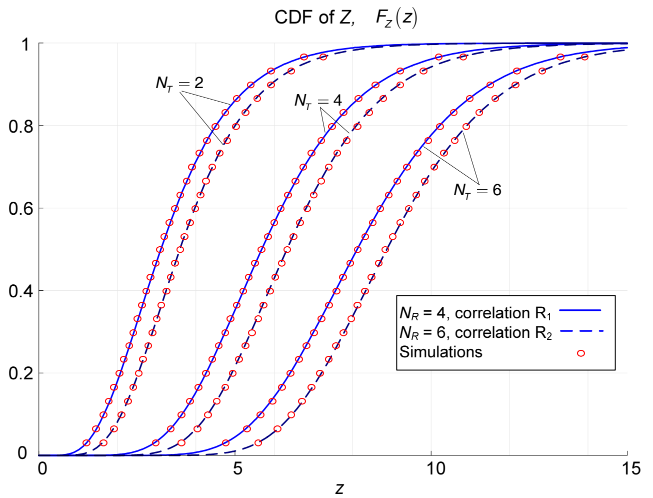

Fig. 1. CDF of Z, the maximum of the square norms of the row vectors of the channel matrix $\mathbf{H}$.

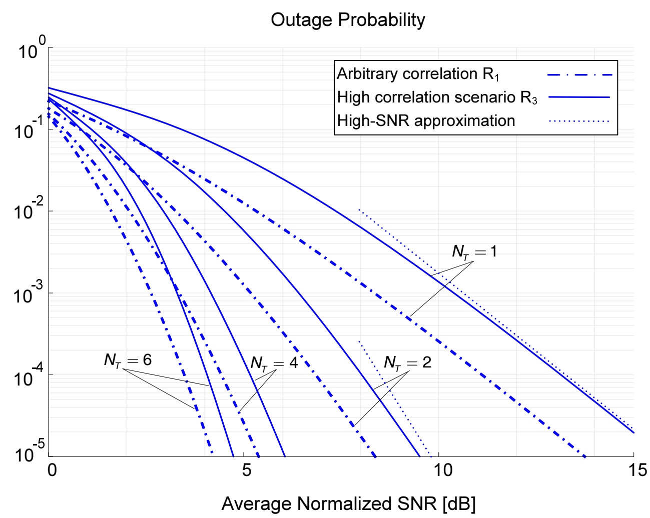

Fig. 2. Outage probability versus average normalized SNR for a SC system with different antenna configurations and arbitrary correlation matrices.

where the coefficients $\hat{c}\left(p_{i,1}^{(n)}, \ldots, p_{i,L}^{(n)}\right)$ depend on N_R, w, and the correlation matrix $\mathbf{R}_{bf}$, and are easily computed with the algorithm in Appendix B. Then, the previous result allows obtaining the outage probability of the codebook-based transmit beamforming system, which is given by

$$P_{out}(x) \triangleq \Pr\{\gamma \leq \gamma_0\} = \Pr\left\{Z \leq \frac{\gamma_0}{\bar{\gamma}}\right\} = F_Z\left(\frac{1}{x}\right) = \sum_{n=0}^{\infty}\sum_{i=1}^{s_n} \hat{c}\left(p_{i,1}^{(n)}, \ldots, p_{i,L}^{(n)}\right) \prod_{j=1}^{L} \Delta_{p_{i,j}^{(n)}}^{N_R}\left(\frac{w^2}{x}\right). \tag{46}$$

V. NUMERICAL RESULTS

The derived expressions have been numerically evaluated in order to analyze the performance of such systems. To check the validity of the derived expressions, we also provide some Monte-Carlo simulation results for the CDF of the SNR at the output of the combiner. Results in Figs. 1 and 2 correspond to the performance of the MIMO system with SC under arbitrarily correlated fading. On the one hand, Fig. 1 shows the CDF of Z which is a scaled version of the SNR at the output of the combiner. The analytical expression in (39) has been evaluated for two different arbitrary correlation matrices, $\mathbf{R}_1$ and $\mathbf{R}_2$ (see Table I), corresponding to $N_R = 4$ and $N_R = 6$ receive antennas respectively, $w = 1$, and several values of N_T. The truncation limit of the series in (39) has been set to $N_{max} = 8$, which yields a total of 25 terms for $N_R = 4$ and 36 terms for $N_R = 6$. It is emphasized that the computation of the corresponding coefficients by the algorithm provided in Appendix B takes less than one second in a common PC. The simulation values of the CDF are also superimposed to the analytical curves in Fig. 1 showing that, with just a few terms of the series, they are nearly in perfect agreement. The rapid convergence of the series is illustrated in Table II, which presents the truncation limit and number of terms needed to achieve 2, 3, and 4 significant figure accuracy. On the other hand, the outage probability of the SC system with $N_R = 4$ is plotted in Fig. 2 for different values of N_T and two correlation matrices. In this case, the high-correlation matrix proposed in [30] for a typical microcell scenario ($\mathbf{R}_3$), is compared to the low-medium correlation scenario defined by the arbitrary matrix $\mathbf{R}_1$. The scale factor has been set to $w = 0.93$ for $\mathbf{R}_3$ to assure the convergence of the series. Also, the high-SNR approximation for the outage probability has been plotted for $\mathbf{R}_3$ and $N_T = 1, 2$. It is observed that the high-correlation scenario determined by $\mathbf{R}_3$ degrades significantly the performance with respect to the arbitrary $\mathbf{R}_1$.

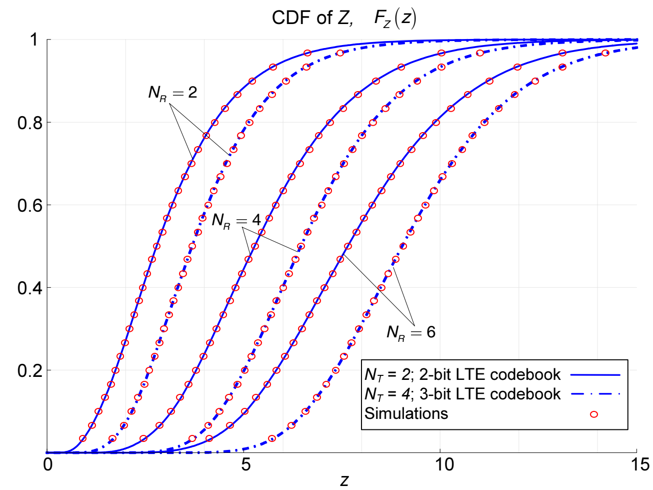

Fig. 3. CDF of Z, the effective SNR at the output of the MRC processing when the optimal beamforming codeword is employed.

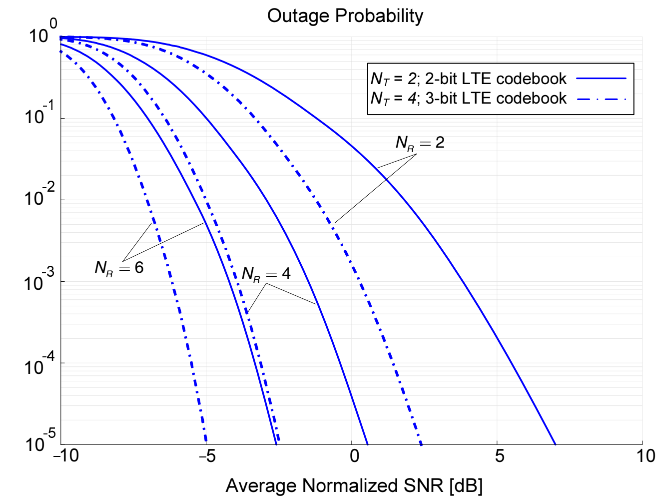

Fig. 4. Outage probability versus average normalized SNR for a beamforming-MRC system with different antenna configurations and LTE-based codebooks.

TABLE II
TRUNCATION LIMIT (N_{max}) AND NUMBER OF TERMS NEEDED IN (39) ($N_T = 4$) TO ACHIEVE 2, 3, AND 4 SIGNIFICANT FIGURE ACCURACY.

		$N_R = 4$ ($\mathbf{R}_1$)			$N_R = 6$ ($\mathbf{R}_2$)		
		$Z=5$	$Z=7.5$	$Z=10$	$Z=5$	$Z=7.5$	$Z=10$
2-Fig	N_{max}	2	2	2	2	2	2
	Terms	2	2	2	2	2	2
3-Fig	N_{max}	4	4	2	4	2	2
	Terms	6	6	2	6	2	2
4-Fig	N_{max}	8	6	4	8	6	4
	Terms	25	13	6	36	16	6

However, it is shown that the performance loss associated to the high-correlation scenario can be reduced by increasing the number of transmit antennas.

Analogously, Figs. 3 and 4 depict the performance results for the codebook-based transmit beamforming system with MRC at the receiver. The system performance has been evaluated for three different values of the receive antennas, $N_R = 2, 4, 6$, and two transmit antenna configurations $N_T = 2, 4$ with 2-bit and 3-bit codebooks respectively. These codebooks have been chosen according to the Long Term Evolution (LTE) cellular technology specifications [31]. Fig. 3 shows the numerical evaluation of the CDF of Z, given in (45), for the mentioned cases. The simulated CDF has also been superimposed to the analytical curves in order to check the validity of the derived expression. In this case, the truncation limit has been set to $N_{max} = 10$ yielding a total of 20 and 37 terms for the 2-bit and the 3-bit codebooks, respectively. Again, it is observed that the simulated values fit reasonably well with the analytical ones for the considered series truncation limit. The outage probability of such system is shown in Fig. 4 for the same antenna configurations and codebooks.

VI. CONCLUSION

In this paper, we have derived the joint PDF and CDF of the diagonal elements of a complex Wishart matrix, which is a particular multivariate chi-square distribution. The CDF expression is in the form of an infinite series representation in terms of the well-known Laguerre polynomials, and has been shown to be easily computable. This expression has been used to obtain the distribution of the maximum of the diagonal elements, which allows analyzing the performance of two different MIMO systems under practical conditions. First, our statistical results have been applied to the outage probability analysis of MIMO systems with receive antenna selection in arbitrarily correlated Rayleigh fading. Then, the same analytical approach has been applied to obtain an expression for the outage probability of codebook-based transmit beamforming systems with MRC at the receiver.

APPENDIX A
PROOF OF PROPOSITION 1

The coefficients $c(n_1, \ldots, n_k)$ can be obtained from the expansion of the CF in (5). Specifically, the expansion

$$|\mathbf{I} - (\mathbf{I} - \mathbf{WRW})\,\mathbf{U}|^{-p} = \sum_{n=0}^{\infty} \underbrace{\sum_{(n=n_1+\cdots+n_k)} c(n_1, \ldots, n_k) \prod_{j=1}^{k} u_j^{n_j}}_{\theta_n(u_1, u_2, \ldots, u_k)} \tag{47}$$

has been used in the derivation of (9). The determinant in (47) can be expressed as [23]

$$|\mathbf{I} - (\mathbf{I} - \mathbf{WRW})\,\mathbf{U}| = 1 + \sum_{j=1}^{k} D_j, \tag{48}$$

where D_j is the j-order polynomial defined in (16). Now, considering (48) and making use of the binomial expansion, it is possible to write

$$|\mathbf{I} - (\mathbf{I} - \mathbf{WRW})\,\mathbf{U}|^{-p} = \sum_{n=0}^{\infty} \binom{p+n-1}{n} (-1)^n \left(\sum_{j=1}^{k} D_j\right)^n. \tag{49}$$

Then, after applying the multinomial theorem [26, (24.1.2)] to (49) and further simplifications the expansion can be rewritten as

$$\left|\mathbf{I}-\left(\mathbf{I}-\mathbf{WRW}\right)\mathbf{U}\right|^{-p}= \sum_{n=0}^{\infty}\frac{\Gamma\left(p+n\right)}{\Gamma\left(p\right)}\sum_{(n=\ell_1+\ldots+\ell_k)}\prod_{j=1}^{k}\frac{\left(-D_j\right)^{\ell_j}}{\ell_j!}. \tag{50}$$

Finally, the expression (15) for the polynomial $\theta_n\left(u_1,u_2,\ldots,u_k\right)$ with coefficients $c\left(n_1,\ldots,n_k\right)$ is obtained by rearranging the terms of the expansion in (50).

APPENDIX B MATHEMATICA™ 7.0 PROGRAM FOR THE COEFFICIENTS COMPUTATION

```
Needs["Combinatorica`"]
GetCoefs[R_, nDeg_, w_, Nmax_] := Module[{Pcoef, Vcoef},

(* INITIALIZATION *)
nVar = Length[R]; Nr = nDeg;
Pcoef = List[ConstantArray[0, nVar]]; Vcoef = List[1];
W = w*IdentityMatrix[nVar];
Rp = IdentityMatrix[nVar] - W.R.W;
Pos = Table[KSubsets[Range[nVar], j], {j, nVar}];
Val = Table[A=Pos[[j]]; Table[Re[Det[Rp[[A[[i]], A[[i]]]]]],
  {i, Length[A]}], {j, nVar}];

(* AUX. ROUTINE: INTEGER SOL. OF l1 + 2 l2 +...+ k lk = n *)
Sol[K_] := Module[{r, l},
  Clear[n];
  cond = Table[If[Apply[And, PossibleZeroQ[Val[[i]]]],
    n[i] == 0, n[i] >= 0], {i, 1, nVar}];
  r = Reduce[Join[{K == Sum[j n[j], {j, nVar}]}, cond],
    Table[n[i], {i, nVar}], Integers];
  l = List[ToRules[r]];
  Table[n[i], {i, nVar}] /. l];

(* AUX. ROUTINE: COMPUTATION OF POLYNOMIAL Theta_n *)
Polynomial[N_] := Module[{poly = 0},
  Y = Table[y[i], {i, nVar}];
  If [w == 1, Dr[1] := 1;, Clear[Dr];];
  Dr[r_] := (-1)^r Sum[Val[[r]][[m]] Product[
    Y[[Pos[[r]][[m]][[l]]]],{l, r}], {m, Length[Val[[r]]]}];
  intSol = Sol[N];
  If[ArrayDepth[intSol] == 1, ,
    poly = (Gamma[Nr])^-1 Sum[Gamma[Nr + Total[intSol[[m]]]]
      Product[(-Dr[l])^intSol[[m]][[l]]/(intSol[[m]][[l]])!,
        {l, nVar}], {m, Length[intSol]}]]];

(* GET COEFFICIENTS c^, defined under (24), FOR n=1:Nmax *)
For[m = 1, m < Nmax + 1, m++,
  polym = Polynomial[m]; Clear[a, p, t];
  If[Length[polym] == 0, ,
    a = CoefficientRules[polym]; b = List[];
    For[k = 1, k < Length[a] + 1, k++,
      AppendTo[b, Sort[a[[k]][[1]],
        Greater] -> a[[k]][[2]]];];
    b = Sort[b]; p = b[[1]][[1]]; t = b[[1]][[2]];
    AppendTo[b, ConstantArray[0, nVar] -> 1];
    For[l = 2, l < Length[a] + 2, l++,
      If[b[[l]][[1]] == p, t = t + b[[l]][[2]];,
        AppendTo[Pcoef, p];
        AppendTo[Vcoef, t];
        t = b[[l]][[2]]; p = b[[l]][[1]];];];];];
        {Pcoef, Vcoef}]
```

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