

Outage Probability Analysis for η - μ Fading Channels

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Abstract—In this Letter we derive exact closed-form expressions for the outage probability (OP) in η - μ fading channels. First, a general expression in terms of the confluent Lauricella function is derived for arbitrary values of μ . Next, we restrict the analysis to physical η - μ channel models, i.e. to integer values of 2μ , and obtain exact closed-form expressions for the OP in terms of Marcum Q, Bessel and elementary functions. The results in this Letter are applicable to the OP analysis of maximal ratio combining (MRC) over i.i.d. η - μ or Hoyt fading channels.

Index Terms— η - μ fading, Nakagami- q (Hoyt) fading, outage probability, maximal ratio combining.

I. INTRODUCTION

THE η - μ fading model considers a general non-line-of-sight (NLOS) propagation scenario. By setting two shape parameters η and μ , this model includes some classical fading distributions as particular cases, e.g. Nakagami- q (Hoyt), One-Sided Gaussian, Rayleigh and Nakagami- m . It has been shown that the fit of the η - μ distribution to experimental data is better than the classical distributions previously mentioned. A detailed description of the η - μ fading model can be found in [1] and references therein.

The outage probability (OP) is a key performance metric in wireless communications subject to fading. Given the recent relevancy of η - μ fading channels, obtaining analytical expressions for the OP is particularly interesting. To the best of authors' knowledge, exact and closed-form expressions for the OP in η - μ fading channels are not found in the literature.

The outage probability can be easily obtained from the cumulative distribution function (CDF) of power η - μ random variables. Besides, thanks to the reproductive property of the η - μ distribution [1], the extension of the OP analysis to maximal ratio combining (MRC) is trivial. Moreover, it was shown in [2] that the η - μ distribution is an accurate approximation to the sum of i.n.i.d. Nakagami- q (Hoyt) random variables. Hence, the η - μ power CDF also allows to analyze the OP of MRC over i.n.i.d. Hoyt fading channels. The reasons above motivate us to focus on the CDF of power η - μ random variables. In [1], Yacoub defined the following integral to represent the complement of the CDF of the power η - μ fading distribution

$$Y_\mu(x, y) \triangleq \frac{\sqrt{\pi} 2^{\frac{3}{2}-\mu} (1-x^2)^\mu}{\Gamma(\mu) x^{\mu-\frac{1}{2}}} \int_y^\infty e^{-t^2} t^{2\mu} I_{\mu-\frac{1}{2}}(t^2 x) dt, \quad (1)$$

where $-1 < x < 1$ and $y \geq 0$.

Manuscript received December 31, 2009. The associate editor coordinating the review of this letter and approving it for publication was G. K. Karagiannis.

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This work is partially supported by the Spanish Government under project TEC2007-67289/TCM and by AT4 wireless.

Digital Object Identifier 10.1109/LCOMM.2010.06.092501

In this Letter, we derive closed-form expressions for Yacoub's integral $Y_\mu(x, y)$ which allow us to obtain new closed-form expressions for the OP in η - μ fading channels. Two types of analytical results are obtained for either arbitrary fading or physical fading models. First, we obtain a general expression in terms of the confluent Lauricella function $\Phi_2^{(2)}$ [3, eq. 5.71.21][4] for arbitrary η - μ fading channels. Next we consider physical fading models, i.e. those with an integer number of multipath clusters $N = 2\mu$, and show that Y_μ can be integrated in terms of the classical Marcum Q and Bessel functions. Then, the results on Yacoub's integral are applied to obtain novel analytical expressions for the OP in η - μ fading channels.

The remainder of this paper is organized as follows. The closed-form expressions for Yacoub's integral are presented in Section II. In Section III we apply these results to compute the outage probability in η - μ fading channels. Finally, some conclusions are given in Section IV.

II. CLOSED-FORM EXPRESSIONS FOR YACOUB'S INTEGRAL

A. Arbitrary η - μ distribution

The η - μ fading distribution is fully characterized in terms of measurable physical parameters. Therefore, it is possible to fit experimental data by adequately setting the two shape parameters η and μ . In the following proposition we derive a general expression for (1), which is valid for an arbitrary value of the μ parameter, i.e. for a real positive μ .

Proposition 1: Yacoub's integral Y_μ defined in (1) can be expressed as

$$Y_\mu(x, y) = 1 - \frac{(1-x^2)^\mu y^{4\mu}}{\Gamma(1+2\mu)} \times \Phi_2(\mu, \mu; 1+2\mu; -(1+x)y^2, -(1-x)y^2), \quad (2)$$

where $\Phi_2 \equiv \Phi_2^{(2)}$ is the confluent Lauricella function [3, eq. 5.71.21][4].

Proof: See Appendix I. ■

B. Physical η - μ distribution

Now we restrict the analysis to the case of physical channel models, which assume an integer number of multipath clusters. In this case, only a multiple of 1/2 is allowed for the μ parameter, being $N = 2\mu$ the number of clusters. In the subsequent, it is shown that Y_μ with an integer value of 2μ can be expressed in terms of classical functions within the communications theory context.

For convenience, a special function associated to I_m is introduced in the following definition, where I_m is the m th-order modified Bessel function of the first kind.

Definition 1 (Incomplete Lipschitz-Hankel integral of I_m):

$$I_{e_m}(x; \alpha) \triangleq \int_0^x t^m e^{-\alpha t} I_m(t) dt, \quad (3)$$

where $\alpha \in \mathbb{R}$, $m \in \mathbb{N}$; $\alpha > 1$ and $x \in [0, \infty)$.

The incomplete Lipschitz-Hankel integrals (ILHI), studied by Agrest and Maksimov [5], are functions of interest in several areas of science and engineering. Recent results in [6] show that the function in Definition 1 can be expressed in closed-form by means of a finite combination of Marcum Q and Bessel functions.

Proposition 2: The m th-order ILHI I_{e_m} is expressed as

$$I_{e_m}(x; \alpha) = A_{m\alpha}^0(\alpha) + A_{m\alpha}^1(\alpha) Q_1\left(\frac{\sqrt{x}}{\sqrt{\alpha + \sqrt{\alpha^2 - 1}}}, \sqrt{x} \sqrt{\alpha + \sqrt{\alpha^2 - 1}}\right) + e^{-\alpha x} \sum_{i=0}^m \sum_{j=0}^{m+1} B_m^{i,j}(\alpha) x^i I_j(x), \quad (4)$$

where Q_1 is the first-order Marcum Q function. The coefficients $A_m^l(\alpha)$, $B_m^{i,j}(\alpha)$ can be obtained recursively in a finite number of steps after identifying $A_m^l(\alpha) = \mathcal{A}_{m,m}^l(\alpha)$ and $B_m^{i,j}(\alpha) = \mathcal{B}_{m,m}^{i,j}(\alpha)$ in the algorithm given in [6, Appendix III].

Proof: See [6, Appendix III]. ■

The previous result allows obtaining an exact and closed-form expression for the Yacoub's integral Y_μ . For convenience in the derivation of this expression, we consider two different cases for either an odd or even number of multipath clusters N , i.e. for either half-integer or integer values of μ . On the one hand, if μ is a half-integer, the integral Y_μ can be rewritten in terms of the ILHI introduced in Definition 1. Then, the result from Proposition 2 may be applied to obtain Y_μ in closed-form. On the other hand, for integer values of μ the final Y_μ expression is given by a linear combination of elementary functions involving polynomials and exponentials.

The following proposition gathers the two derived expressions for the integral Y_μ , which are key results for the final analysis of the outage probability.

Proposition 3: For integer values of 2μ , the Yacoub's integral Y_μ defined in (1) can be expressed by one of the two following formulas. If 2μ is odd, i.e. μ is a half-integer, Y_μ is expressed as

$$Y_\mu(x, y) = 1 - \frac{2^{\frac{1}{2}-\mu} \sqrt{\pi} (1-x^2)^\mu}{|x|^{2\mu} \Gamma(\mu)} I_{e_{\mu-\frac{1}{2}}}\left(|x|y^2; \frac{1}{|x|}\right). \quad (5)$$

Otherwise, when 2μ is even, the Yacoub's integral is calculated as

$$Y_\mu(x, y) = 1 - (1-x^2)^\mu \left\{ \frac{1}{(1+x)^\mu (1-x)^\mu} + \sum_{k=0}^{\mu-1} \frac{(-1)^{\mu-1} (1-x)^k 2^{-\mu-k}}{\Gamma(\mu-k) (1+x)^{k+1}} x^{-\mu-k} y^{2\mu-2-2k} \times e^{-y^2(1+x)} P_k^{(-1-k, -\mu-k)}\left(\frac{3x+1}{x-1}\right) + \sum_{k=0}^{\mu-1} \frac{(-1)^{k+1} (1+x)^k 2^{-\mu-k}}{\Gamma(\mu-k) (1-x)^{k+1}} x^{-\mu-k} y^{2\mu-2-2k} \times e^{-y^2(1-x)} P_k^{(-1-k, -\mu-k)}\left(\frac{3x-1}{x+1}\right) \right\}, \quad (6)$$

where $P_n^{(a,b)}$ are the Jacobi polynomials [7].

Proof: See Appendix II. ■

Note that, as shown in previous propositions, an exact and closed-form expression can be obtained for Y_μ when 2μ is an integer, i.e. for physical channel models. This result is exploited in the next section, where the exact outage probability is obtained in terms of Marcum Q, Bessel, and elementary functions, thus avoiding the need for numerical integration.

III. OUTAGE PROBABILITY

In this section we connect previous results on Yacoub's integral with the outage probability in η - μ fading channels. Analytical expressions are obtained for the OP and some numerical results are provided. Besides, Monte-Carlo simulation results are presented in order to validate the derived expressions.

Let Ω be the normalized power of the fading signal, i.e. $E[\Omega] = 1$. The instantaneous SNR of the channel is $\gamma \doteq \Omega \bar{\gamma}$, where $\bar{\gamma}$ is the average SNR. Then, after taking into account [1, eq. 19], the outage probability for the η - μ fading channel can be expressed as

$$P_{out}(X) \doteq \Pr\{\gamma \leq \gamma_o\} = 1 - Y_\mu\left(\frac{H}{h}, \sqrt{\frac{2h\mu}{X}}\right), \quad (7)$$

where γ_o is the outage threshold, $X \doteq \frac{\bar{\gamma}}{\gamma_o}$ the average normalized SNR and

$$\begin{cases} h = \frac{2 + \eta^{-1} + \eta}{4}, H = \frac{\eta^{-1} - \eta}{4}, 0 < \eta < \infty \text{ for format 1,} \\ h = \frac{1}{1 - \eta^2}, H = \frac{\eta}{1 - \eta^2}, -1 < \eta < 1 \text{ for format 2.} \end{cases} \quad (8)$$

For details on the two formats of the η - μ distribution the reader is referred to [1]. Now, it is straightforward to obtain the closed-form expressions for the OP by substituting the Yacoub's integral in (7) with the results from previous section. Three different expressions are provided for cases involving an arbitrary, half-integer, or integer value of μ by substituting (2), (5) or (6), respectively, into equation (7). Recall that the first expression involving an arbitrary μ is valid for a general η - μ fading distribution, whereas the second and third expressions correspond to physical channel models with an integer number of multipath clusters ($N = 2\mu$). For brevity, the final OP expressions are not explicitly written here.

The extension of the OP analysis to MRC is straightforward from previous results. Let us consider a MRC receiver with N_r independent branches. Let Ω_n be the normalized power of the fading signal corresponding to the n -th branch. In this case, the effective instantaneous SNR after the MRC processing is given by $\gamma_{MRC} \doteq \bar{\gamma} \sum_{n=1}^{N_r} \Omega_n$. It can be seen in [1] that the sum of N_r i.i.d. η - μ power variates is also η - μ distributed with parameters η and μN_r . Then, the OP of a MRC receiver can be obtained by substituting μ with μN_r in (7).

Fig. 1 shows some numerical plots from the derived analytical expressions. The OP for format 1 η - μ fading channels is depicted as a function of the average SNR for different values of the η - μ parameters. Results are shown for arbitrary η - μ fading distributions ($\mu = 1.2$) and physical channel models

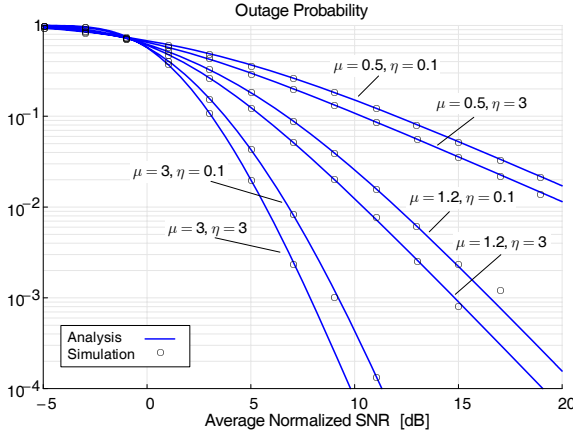


Fig. 1. Outage probability versus normalized average SNR for η - μ fading channels (format 1).

($\mu = 0.5$ and $\mu = 3$). Note that the cases with $\mu = 0.5$ correspond to the Nakagami- q (Hoyt) distribution with $q^2 = \eta$. Specifically, the analytical results for $\mu = 1.2$, $\mu = 0.5$, and $\mu = 3$ are obtained from (2), (5) and (6), respectively. Simulation results are also superimposed to the analytical curves, being shown that they are in perfect agreement. Note that, though numerical results are omitted for brevity reasons, the analysis is also valid for format 2 fading channels.

IV. CONCLUSIONS

Three different closed-form expressions have been derived for the outage probability in η - μ fading channels. For arbitrary values of the parameter μ the OP is expressed in a compact form using the confluent Lauricella function $\Phi_2^{(n)}$. For physical channel models with an odd number of multipath clusters (2μ), the OP is given by Marcum Q and Bessel functions, whereas in case of an even 2μ the analytical results are in terms of elementary functions. Our analytical results are also applicable to compute the OP of MRC receivers in i.i.d. η - μ fading channels.

APPENDIX A

PROOF OF PROPOSITION 1

After making the change of variable $z = xt^2$ in (1), we can write

$$Y_\mu(x, 0) = \frac{2^{\frac{1}{2}-\mu} \sqrt{\pi} (1-x^2)^\mu}{x^{2\mu} \Gamma(\mu)} \int_0^{\pm\infty} z^{\mu-\frac{1}{2}} e^{-\frac{z}{x}} I_{\mu-\frac{1}{2}}(z) dz, \quad (9)$$

where the sign of the upper integration limit is in accordance with the sign of x . Then, by making use of [5, eq. 5.7] we check that $Y_\mu(x, 0) = 1$. From this fact, and considering the previous change of variable, we can express (1) as

$$Y_\mu(x, y) = 1 - \frac{2^{\frac{1}{2}-\mu} \sqrt{\pi} (1-x^2)^\mu}{x^{2\mu} \Gamma(\mu)} \times \int_0^{xy^2} z^{\mu-\frac{1}{2}} e^{-\frac{z}{x}} I_{\mu-\frac{1}{2}}(z) dz. \quad (10)$$

Denoting $p = xy^2$ and $q = 1/x$ and with the help of [8, eq. 29.3.50], we can rearrange the Laplace transform $\mathcal{L}[f(p); p, s] \doteq \int_0^\infty f(p) e^{-ps} dp$ of the integral in (10) in the following form

$$\begin{aligned} \mathcal{L} \left[\int_0^p z^{\mu-\frac{1}{2}} e^{-qz} I_{\mu-\frac{1}{2}}(z) dz; p, s \right] &= \\ \frac{2^{\mu-\frac{1}{2}} \Gamma(\mu)}{\sqrt{\pi}} \frac{1}{s(s+q+1)^\mu (s+q-1)^\mu} &= \\ \frac{2^{\mu-\frac{1}{2}} \Gamma(\mu)}{\sqrt{\pi}} \frac{1}{(q^2-1)^\mu} \frac{1}{s \left(1 + \frac{s}{q+1}\right)^\mu \left(1 + \frac{s}{q-1}\right)^\mu} &= \\ \frac{2^{\mu-\frac{1}{2}} \Gamma(\mu)}{\sqrt{\pi}} \frac{1}{(q^2-1)^\mu} \left\{ \frac{(q^2-1)^\mu}{\Gamma(1+2\mu)} \right\} & \\ \times \left\{ \frac{\Gamma(1+2\mu)}{s^{1+2\mu}} \right\} \left(1 - \frac{-(q+1)}{s}\right)^{-\mu} \left(1 - \frac{-(q-1)}{s}\right)^{-\mu} &. \end{aligned} \quad (11)$$

Identifying [9, eq. 3.43.1.4] with (11) and substituting in (10) we obtain the desired result.

APPENDIX B

PROOF OF PROPOSITION 3

The result for an odd 2μ is obtained after identifying Definition 1 in (10) and taking into account the sign of x . The result for an even 2μ is again obtained working out the integral in (10). First, [8, eq. 29.3.50] is used to represent the involved integral by an inverse Laplace transform

$$\begin{aligned} \mathcal{L} \left[\int_0^p z^{\mu-\frac{1}{2}} e^{-qz} I_{\mu-\frac{1}{2}}(z) dz; p, s \right] &= \\ \frac{2^{\mu-\frac{1}{2}} \Gamma(\mu)}{\sqrt{\pi}} \frac{1}{2\pi j} \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} \frac{e^{sx}}{s(s+q+1)^\mu (s+q-1)^\mu} ds. & \end{aligned} \quad (12)$$

Then, since μ is a nonnegative integer in this case, we can use the well-known residue theorem and, after some tedious but straightforward algebra, the desired expression is obtained.

REFERENCES

- [1] M. D. Yacoub, "The κ - μ and the η - μ distribution," *IEEE Antennas and Propag. Mag.*, vol. 49, pp. 68-81, Feb. 2007.
- [2] J. C. S. Santos Filho and M. D. Yacoub, "Highly accurate η - μ approximation to sum of M independent non-identical Hoyt variates," *IEEE Antennas and Wireless Propag. Lett.*, vol. 4, pp. 436-438, Dec. 2005.
- [3] A. Erdelyi, *High Order Transcendental Functions*, vol. 1. McGraw-Hill, 1953.
- [4] H. Exton, *Multiple Hypergeometric Functions and Applications*. John Wiley & Sons (Halsted Press), 1976.
- [5] M. M. Agrest and M. Z. Maksimov, *Theory of Incomplete Cylindrical Functions and their Applications*. New York: Springer-Verlag, 1971.
- [6] J. F. Paris, E. Martos-Naya, U. Fernández-Plazaola, and J. López-Fernández, "Analysis of adaptive MIMO beamforming under channel prediction errors based on incomplete Lipschitz-Hankel integrals," *IEEE Trans. Veh. Technol.*, vol. 58, pp. 2815-2824, July 2009.
- [7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. San Diego: Academic Press, 2000.
- [8] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, 9th ed. New York: Dover, 1970.
- [9] A. P. Prudnikov and Y. A. Brychkov, *Integrals and Series*, vol. 4. Gordon and Breach Science Publishers, 1992.